# Triangularizing Matrices by Congruence

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## 1. INTRODUCTION

This paper is concerned with the triangularization by congruence of general (square) matrices over a fairly general field. Section 2 answers (over any field except $GF(2)$) the question: Which matrices are congruent to triangular matrices? The answer is: All matrices except the nonzero skew matrices. Sections 3 and 4 find (over certain ordered fields, in particular, over the real field) for a given congruence class the maximum number of positive diagonal entries a triangular matrix of that class can have. (With certain obvious exceptions, this maximum number is the rank or one less than the rank, depending on the given congruence class.) Section 5 mentions (mostly without proof) various extensions of the results in the earlier sections.

The idea of triangularizing a (nonsymmetric and nonskew) matrix by congruence is relatively untainted by applications. (The present author, however, recently came across an application in which the Corollary at the end of Section 5 of this paper supplied the nontrivial part of a crucial proof. That Corollary may be regarded as the goal toward which, with some digressions, the present paper is directed.) The only standard reference (known to this author) which even purports to treat the problem of triangularizing a general (square) matrix by congruence is [6, Chapter VII, Section 9, pp. 94–95]. That treatment contains an error, whose persistence through several reprintings can be explained best by the above-mentioned lack of applications. (One can assume that [6, loc. cit.] was really intended to treat triangularization by *conjunctivity*, since—over any field, including, with proper definitions and proper modifications

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in the proof, any field of characteristic 2—this section would be entirely correct if the word "congruent" were replaced throughout by the word "conjunctive.")

Throughout the present paper we shall denote by $F$ the scalar field over which all matrices and congruences are taken ($F$ is to be understood when not mentioned explicitly). $F$ is to be regarded as fixed, except where we indicate otherwise (e.g., where, as in the next paragraph, we point out the dependence on $F$ of a definition). In this section $F$ is perfectly arbitrary, but in later sections we shall introduce various restrictions on $F$.

For purposes of this paper "triangularizable over $F$" will mean "congruent over $F$ to a triangular matrix." Thus the triangularizability of a matrix appears to depend on which field (containing its entries) is chosen. However, we shall see in the next section that "triangularizability over $F$" is actually independent of $F$ except when $F = GF(2)$.

In order to avoid misunderstanding, we shall be explicit about which definition for "skew" (short for "skew-symmetric") we are using. (There are two standard definitions, which are not equivalent over fields of characteristic 2 but are equivalent over all other fields.) We say a (square) matrix $S$ is *skew* provided $S$ has zero diagonal and $S + S' = 0$ (when $S$ is a matrix, we shall always use $S'$ to denote the transpose of $S$), or, equivalently, provided the quadratic form corresponding to $S$ vanishes identically.

Now, it is well known that rank and skew-symmetry are preserved by congruence, so a nonzero skew matrix can be congruent only to a nonzero skew matrix. Since a nonzero skew matrix cannot itself be triangular, it thus cannot be congruent to a triangular matrix either (this shows where [6, loc. cit.] is in error, since it says in effect that *every* square matrix is congruent to a triangular matrix):

*Fact* 1.1. If $S$ is a nonzero skew matrix, then $S$ is not triangularizable.

(Theorem 1 will give the converse of Fact 1.1, except over $GF(2)$.)

We shall always denote the rank of $S$ by $r(S)$, or sometimes by $r$ when $S$ is understood. We shall denote the $(i, j)$th entry of $S$ (or $T$, etc.) by $S_{ij}$ (or $T_{ij}$, etc.). We shall denote by $S_1$ the (principal) submatrix of $S$ complementary to the first diagonal entry ($= S_{11}$), and give a corresponding meaning to $T_1$. (Thus $S_1$ is the submatrix of $S$ obtained by deleting the first row and first column of $S$.)

We shall need the following elementary fact.

*Fact* 1.2. Let $S$ be an $n \times n$ matrix and let $1 \leqslant h \leqslant n$. Suppose that the first $h - 1$ rows of $S$ are zero and that $S_{h1} \neq 0$. Let $C$ be the $n \times n$ matrix defined by

$$C_{1j} = - S_{h1}^{-1} S_{hj} \qquad \text{for} \quad j = 2, 3, \ldots, n$$

and otherwise $C_{ij} = \delta_{ij}$ (where $\delta_{ij}$ is the Kronecker symbol [3, p. 2]). Let $T = C'SC$. Then $C$ is nonsingular (hence $T$ is congruent to $S$) and the first $h - 1$ rows of $T$ are zero and $T_{h1} = S_{h1}$ and $T_{hj} = 0$ for all $j \geqslant 2$.

*Remark.* When $h = 1$, Fact 1.2 is a formal expression of what in [6, loc. cit.] is called "semi-isolating" $S_{11}$ in $S$.

Throughout this paper we shall be dealing mainly with a special type of triangular matrix. Both the type and the name that we shall use for it were suggested by [6, loc. cit.].

*Definition.* A (square) matrix $S$ will be called *trapezoidal* provided it is lower triangular and its first $r$ diagonal entries are nonzero (where $r = r(S) = \text{rank}\, S$). (Hence, as is pointed out in [6, loc. cit.], the $j$th column of a trapezoidal matrix $S$ is zero if $j > r(S)$.)

It is clear that if $S$ is trapezoidal then $S_1$ is trapezoidal and if also $S_{11}$ is nonzero then $r(S_1) = r(S) - 1$. (It is also clear that the preceding sentence remains valid if "trapezoidal" is replaced throughout by "lower triangular.")

We shall (for purposes of this paper) call a square matrix $S$ *regular* provided its right null space equals its left null space, i.e., provided its right null space equals that of its transpose, i.e., provided

$$SX = 0 \quad \text{ if and only if } \quad S'X = 0.$$

We shall call a square matrix *irregular* provided it is not regular. Thus, for example, a matrix is regular if it is symmetric, skew, or nonsingular. (Hence, if $S$ is an irregular $n \times n$ matrix then $r(S) < n$.) It is clear that regularity and irregularity are congruence-invariant concepts, i.e., if $S$ is regular then every matrix congruent to $S$ is regular. It is also clear that a trapezoidal matrix is regular if and only if each of its rows which contain zero diagonal entries is zero. Hence it is clear that if $S$ is regular and trapezoidal then $S_1$ is regular.

The properties of regularity in the foregoing paragraph all follow immediately, or almost so, from definition. The next property is less immediate, but its proof is a routine exercise in elementary matrix theory and so will be omitted in the present treatment (the uniqueness part is a special case of a result known as Witt's theorem [5, pp. 222–223]):

*Fact* 1.3. Let $S$ be a regular $n \times n$ matrix of rank $r \geqslant 1$. Then there are a nonsingular $n \times n$ matrix $C$ and a nonsingular $r \times r$ matrix $A$ such that (in block form)

$$C'SC = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

and the congruence class of $A$ is uniquely determined by (that of) $S$.

We shall follow, with certain modifications, the notation in [3, Chapter I, Section 2.2, pp. 10–11] in designating submatrices. Namely, let $S$ be an $n \times n$ matrix and let $M$ and $N$ be subsets of the set $\{1, 2, \ldots, n\}$ and let $M'$ and $N'$ be their respective complementary subsets. Then we shall denote by $S[M|N]$ the submatrix of $S$ coming from (or lying in) those rows whose indices are in $M$ and those columns whose indices are in $N$. We shall also sometimes use the expression "the $[M|N]$ submatrix of $S$" in place of $S[M|N]$. (We depart from [3, loc. cit.] in that we always use $S[M|N]$ to denote a *submatrix of* $S$, regardless of the order in which the elements of $M$ and of $N$ are listed. Thus, for example, in our notation,

$$S[2, 1|4, 4] = \begin{bmatrix} S_{14} \\ S_{24} \end{bmatrix} = S[1, 2|4].)$$

We then follow [3, loc. cit.] in the other three ways of designating submatrices: $S[M|N) = S[M|N']$, $S(M|N] = S[M'|N]$, and $S(M|N) = S[M'|N']$. We shall abbreviate further in the case of principal submatrices: $S[M] = S[M|M]$ and $S(M) = S(M|M)$. We shall also sometimes use "the $[M|N)$ submatrix of $S$" for $S[M|N)$, "the $(M)$ submatrix of $S$" for $S(M)$, etc. Thus, in our earlier notation (which we shall continue to use), $S_1 = S(1) = S[2, 3, \ldots, n] =$ the (1) submatrix of $S$.

Using the above notation, we next introduce some terminology concerning congruences. A nonsingular matrix $C$ will be said to *define* the congruence

$$S \to C'SC$$

(and we shall use such expressions as "applying this congruence to $S$ gives $C'SC$" and "$C'SC$ is obtained from $S$ by this congruence"). By the *order* of a congruence we shall mean the order of any of its defining matrices. Now let $S$, $n$, $M$, and $M'$ be as in the last paragraph, let $m$ be the cardinal of $M$, and let $D$ be an $m \times m$ nonsingular matrix. Then by "the $[M]$ *subcongruence* of order $n$ defined by $D$" or "the $[M]$ *subcongruence* of order $n$ corresponding to the congruence (of order $m$) defined by $D$" we shall mean the congruence (of order $n$) defined by the $n \times n$ matrix $C$ satisfying: $C[M] = D$, $C[M|M) = 0$, $C(M|M] = 0$, and $C(M) = I$ (= the identity matrix of order $n - m$). An "$(M)$ subcongruence" will mean the corresponding $[M']$ subcongruence (defined by a matrix of order $n - m$). (We shall usually not specify the order of a subcongruence when it is clear from context.) Thus, as examples, a (1) subcongruence of order $n$ is always a $[2, 3, \ldots, n]$ subcongruence of order $n$ and conversely; and a $[2, 5]$ subcongruence is always a $[2, 4, 5]$ subcongruence, but not conversely.

When $i \neq j$, the $[i, j]$ subcongruence (of order $n$) defined by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

will be called "the interchanging $[i, j]$ subcongruence" (of order $n$). (It is the result of interchanging rows $i$ and $j$ and then interchanging columns $i$ and $j$.) "The interchanging $[i, i]$ subcongruence" (of order $n$) will mean the identity congruence (of order $n$).

Basic to all our applications of the idea of subcongruence is the following fact, Fact 1.4. However, in certain applications we shall find several special cases of Fact 1.4 more immediately relevant, so we list three of these special cases (Facts 1.5, 1.6, and 1.7) for later convenience (Fact 1.7 is actually also a special case of Fact 1.6). The proofs of all four facts are completely routine and so will be omitted.

*Fact* 1.4. Let (as above) $S$ be an $n \times n$ matrix, $M$ be a subset of $\{1, 2, \ldots, n\}$, $m$ be the cardinal of $M$, and $D$ be an $m \times m$ nonsingular matrix, and let $T$ be obtained from $S$ by the $[M]$ subcongruence defined by $D$. Then $T[M] = D'S[M]D$, $T[M|M) = D'S[M|M)$, $T(M|M] = S(M|M]D$, and $T(M) = S(M)$. Hence any zero columns of $S[M|M)$ and of $T[M|M)$ correspond, and any zero rows of $S(M|M]$ and of $T(M|M]$ correspond.

*Fact* 1.5. If, in Fact 1.4, $M = \{2, 3, \ldots, n\}$, $[S_{11}] = S(M) \neq 0$, $S[1|1) = S(M|M] = 0$, and $T_1 = T(1) = T[M]$ is trapezoidal, then $T$ is trapezoidal and $T_{11} = S_{11}$.

*Fact* 1.6. Let, in Fact 1.4, $S$ be trapezoidal, $M = \{h + 1, h + 2, \ldots, h + m\}$ be *contiguous*, and $D'S[M]D = T[M]$ be trapezoidal. Then $T$ is trapezoidal and $T(M) = S(M)$. (The result remains valid if "trapezoidal" is replaced throughout by "lower triangular.")

*Fact* 1.7. Let $S$ be trapezoidal, $h$ and $l$ be given such that $r \geqslant h > l$, $S[l + 1, l + 2, \ldots, h \mid l] = 0$, and $S[h \mid l, l + 1, \ldots, h - 1] = 0$. Then the interchanging $[h, l]$ subcongruence applied to $S$ gives a trapezoidal matrix whose diagonal is the same as that of $S$ except for the interchange of the $h$th and $l$th diagonal entries. (The result remains valid if "trapezoidal" is replaced throughout by "lower triangular" and "$r \geqslant$" is deleted.)

We conclude this section with a less elementary result, which will help to justify our preoccupation with trapezoidal matrices.

*Fact* 1.8. A matrix is triangularizable if and only if it is congruent to a trapezoidal matrix.

*Proof.* The "if" part is by definition. The "only if" part obviously follows from the following two facts.

*Fact* 1.8'. Every upper triangular $n \times n$ matrix is congruent to a lower triangular matrix. (Proof: use the congruence defined by the matrix $C$ for which $C_{ij} = \delta_{i, n+1-j}$, where $\delta_{i,k}$ is the Kronecker symbol.)

*Fact* 1.8''. Every lower triangular $n \times n$ matrix is congruent to a trapezoidal matrix.

*Proof of Fact* 1.8''. One proof, if $F \neq GF(2)$, is simply to combine Theorem 1 (next section) with Fact 1.1 above. We present here another proof, which applies whether $F = GF(2)$ or not. We use induction on $n$. The result, Fact 1.8'', is trivial if $n = 1$, so suppose that $n \geqslant 2$ and that every $(n - 1) \times (n - 1)$ lower triangular matrix is congruent to a trapezoidal matrix. Let $S$ be an $n \times n$ lower triangular nonzero matrix (there is nothing to prove if $S = 0$) and let the first nonzero row of $S$ be row $h$.

Let $S_{hk}$ be the first nonzero entry of row $h$. Then the following four cases exhaust the possibilities (since $h \geqslant k \geqslant 1$).

Case 1: $h = 1$. Here $k = 1$ also, i.e., $S_{11} \neq 0$. $S_1$ is a lower triangular $(n-1) \times (n-1)$ matrix so by our induction assertion $S_1$ is congruent to a trapezoidal matrix. By Fact 1.5 the corresponding (1) subcongruence applied to $S$ gives a trapezoidal matrix.

Case 2: $h = k > 1$. Here we apply to $S$ the interchanging $[1, h]$ subcongruence, which by Fact 1.7 gives a matrix which is covered by Case 1.

Case 3: $h > k = 1$. Here, since $S_{hj} = 0$ for $j > h$, we have by Fact 1.2 that $S$ is, by a $[1, 2, \ldots, h]$ subcongruence, congruent to a matrix $T$ whose first $h - 1$ rows are zero and such that $T_{h1} = S_{h1}$ $(\neq 0)$ and $T_{hj} = 0$ for $j \geqslant 2$. Now, by Fact 1.4 we have $T(1, 2, \ldots, h) = S(1, 2, \ldots, h)$, which is lower triangular, so $T$ itself is lower triangular. We next apply to $T$ the $[1, h]$ subcongruence defined by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and get thereby a matrix which is (because of Fact 1.4) covered by Case 1 (and which is congruent to $S$).

Case 4: $h > k > 1$. Here we apply to $S$ the interchanging $[1, k]$ subcongruence and by Fact 1.7 get thus a lower triangular matrix $T$. By Fact 1.4 the first $h - 1$ rows of $T$ are zero and $T_{h1} = S_{hk} \neq 0$, so $T$ is covered by Case 3.

This completes the proof of Fact 1.8″ and hence of Fact 1.8.

## 2. THE GENERAL TRIANGULARIZATION THEOREM

In this section we present our first main result (Theorem 1), the converse (almost) of Fact 1.1. We may regard it as a natural extension of known theorems on the diagonalization by congruence of symmetric nonskew matrices (see [2, Theorem 10, p. 171] if $F$ has characteristic 2 and see [4, Theorem 5-5, p. 90] otherwise).

THEOREM 1. *Let $F$ be any field having at least three elements and suppose that $S$ is an $n \times n$ matrix over $F$ but that $S$ is not a nonzero skew matrix. Then $S$ is congruent over $F$ to a trapezoidal matrix.*

*Proof.* We give a constructive proof, using induction on $n$. The theorem is trivially true if $n = 1$, so suppose that $n \geqslant 2$ and that every

$(n-1) \times (n-1)$ nonskew matrix is congruent (over $F$) to a trapezoidal matrix. Now let $S$ be an $n \times n$ nonskew matrix over $F$ (we have nothing to prove if $S = 0$).

We first show that $S$ is congruent (always over $F$) to a matrix whose first diagonal entry is nonzero. Namely, if $S_{ii} \neq 0$ for some $i$, then we apply to $S$ an interchanging $[1, i]$ subcongruence. Otherwise (that is, if $S_{ii} = 0$ for all $i$), since $S$ is nonskew, there is a pair $(i, j)$ such that $i < j$ and $S_{ij} + S_{ji} \neq 0$. Here we apply to $S$ the $[i, j]$ subcongruence defined by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and get (by Fact 1.4) thus a matrix which has a nonzero diagonal entry (namely, $S_{ij} + S_{ji}$ in the $(i, i)$th place). This is a case dealt with earlier.

Thus in the rest of this proof we may assume $S_{11} \neq 0$. By Fact 1.2 we may further assume $S_{1j} = 0$ for all $j \geqslant 2$. If now $S_1 = 0$ we have no more to prove. If $S_1$ is nonskew, then by our induction assertion $S_1$ is congruent to a trapezoidal matrix (of order $n-1$) and by Fact 1.5 the corresponding (1) subcongruence applied to $S$ gives a trapezoidal matrix.

Thus we assume that $S_1$ is a nonzero skew matrix (as well as that $S_{11} \neq 0$ and $S[1|1) = 0$) and hence that $n \geqslant 3$. By [4, Theorem 5–11, pp. 95–96] (which holds over an arbitrary field) $S_1$ is congruent to a matrix whose $[1, 2]$ submatrix is nonsingular and whose $[1, 2|1, 2)$ submatrix is zero. Applying to $S$ the corresponding (1) subcongruence, we have by Fact 1.4 that $S$ is congruent to a matrix whose $[1, 2, 3]$ submatrix is nonsingular and whose $[1, 2, 3|1, 2, 3)$ submatrix is zero (and whose $[1|2, 3]$ submatrix is zero and whose $[2, 3]$ submatrix is nonzero and skew). Thus we may assume that $S$ itself is in this form, namely, that $S[1, 2, 3|1, 2, 3) = 0$ and that $S[1, 2, 3]$ is of the form

$$A = \begin{bmatrix} a & 0 & 0 \\ u & 0 & -b \\ v & b & 0 \end{bmatrix}$$

where $a \neq 0$ and $b \neq 0$. Now, if $v \neq 0$, we apply to $S$ the $[2, 3]$ subcongruence defined by the (nonsingular) matrix

$$\begin{bmatrix} 0 & -v \\ 1 & u \end{bmatrix}$$

and (in view of Fact 1.4) get thereby a matrix (whose $[1, 2, 3|1, 2, 3)$ submatrix is zero and) whose $[1, 2, 3]$ submatrix is of the same form as $A$ above but with the additional property that its $(3, 1)$st entry is zero. Thus in any case we may assume $v = 0$ in $A$.

Now we choose $t \in F$ so that $t \neq 0$ and $ut \neq b$. (This choice is possible because $F$ has at least three elements and $b \neq 0$.) Then we apply to $S$ the $[1, 2, 3]$ subcongruence defined by the (nonsingular) matrix

$$\begin{bmatrix} 1 & (b - ut)(at)^{-1} & t \\ at(b - ut)^{-1} & 1 & 0 \\ 0 & (b - ut)(at^2)^{-1} & 1 \end{bmatrix},$$

and (in view of Fact 1.4) get thereby a matrix whose $[1, 2, 3]$ submatrix is of the form

$$B = \begin{bmatrix} a(1 - ub^{-1}t)^{-1} & 0 & 0 \\ u_1 & b(b - ut)(at^2)^{-1} & 0 \\ y_1 & x_1 & at^2 \end{bmatrix}$$

and whose $[1, 2, 3|1, 2, 3)$ submatrix is zero. Thus $S$ is congruent to a matrix covered by an earlier case (namely, the case in which $S_1$ was nonskew). This concludes the proof of the induction step and of Theorem 1.

*Remark.* Theorem 1 does not hold if $F = GF(2)$ (= the field having just two elements), as seen from the example

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which, over $GF(2)$, is nonskew and nontriangularizable (since the corresponding quadratic form is nonzero for only two of the eight vectors in its domain). However, one can easily see from the proof of Theorem 1 that, over $GF(2)$, a nonskew matrix is triangularizable if it is symmetric (this is known: see [2, loc. cit.]) or if it has rank $\leqslant 2$. One can also show that every irregular matrix is triangularizable.

## 3. INDEX AND SUPERINDEX

From now on, we assume the field $F$ is ordered. In this section we introduce several definitions and elementary results which will be needed later. Then we consider how these definitions and results apply in certain examples.

We begin with two results (Facts 3.1 and 3.2) which require no new definitions, but only previous definitions plus the new assumption that $F$ is ordered. The first is a corollary to the proof of Theorem 1.

*Fact* 3.1. Let $S$ be an $n \times n$ matrix such that $S + S'$ is not nonpositive definite (i.e., such that $X'(S + S')X$ is positive for at least one $n \times 1$ matrix $X$). Then $S$ is congruent to a trapezoidal matrix whose first diagonal entry is positive.

*Proof.* Since $S + S'$ is symmetric, it is congruent to a diagonal matrix $2D$. Since $S + S'$ is not nonpositive definite, $D$ is also not nonpositive definite [4, Theorems 5–1 and 5–2, pp. 86–87] and so has at least one positive diagonal entry; in fact, we may assume the first diagonal entry of $D$ is positive (as in the proof of [4, Theorem 5–6, p. 91]). Since $S + S'$ is congruent to $2D$, there is a nonsingular $n \times n$ matrix $C$ such that $C'(S + S')C = 2D$. Let $T = C'SC$. Then $T$ is congruent to $S$, and $T + T' = 2D$, so $T_{11} = D_{11} > 0$. Thus $S$ is congruent to a matrix whose first diagonal entry is positive, so we may assume for the rest of this proof that $S_{11} > 0$. Furthermore, by Fact 1.2 we may assume also that $S_{1j} = 0$ for all $j \geqslant 2$.

We now have two cases to consider.

Case 1: $S_1$ is not a nonzero skew matrix. Here $S_1$ is congruent to a trapezoidal matrix by Theorem 1. Applying to $S$ the corresponding (1) subcongruence, we get by Fact 1.5 a trapezoidal matrix which has the same first diagonal entry as $S$ has (namely, $S_{11}$, which is positive).

Case 2: $S_1$ is a nonzero skew matrix. Here, as in the proof of Theorem 1, we may assume (that $S[1, 2, 3 \mid 1, 2, 3) = 0$ and) that $S[1, 2, 3]$ has the form of the matrix $A$ of that proof, with $v = 0$ and $a > 0$ (and $b \neq 0$). Then, as in that proof, $S$ is congruent to a matrix whose $[1, 2, 3]$ submatrix has the form of the matrix $B$ there (and whose $[1, 2, 3 \mid 1, 2, 3)$ submatrix is zero) and which is therefore covered by Case 1 above if we pick a nonzero $t$ so that $1 - ub^{-1}t > 0$ (which obviously we can do in any ordered field). This completes the proof of Fact 3.1.

*Fact* 3.2. If $S$ is an irregular $n \times n$ matrix, then $S + S'$ is indefinite.

*Proof.* Skew matrices are all regular, so by Theorem 1 $S$ is congruent to a trapezoidal matrix. Since indefiniteness is preserved under congruence

[4, Theorems 5–1 and 5–2, pp. 86–87] (as is matrix addition and transposition and irregularity), we may assume that $S$ itself is trapezoidal. Let (as always) $r$ be the rank of $S$. Since $S$ is irregular and trapezoidal, there are an $i > r$ and a $j \leqslant r$ such that $S_{ij} \neq 0$. One easily sees that then the $[j, i]$ submatrix of $S + S'$ is of the form

$$\begin{bmatrix} a & x \\ x & 0 \end{bmatrix}, \qquad x \neq 0$$

(where $a = 2S_{jj}$ and $x = S_{ij} \neq 0$) and is thus indefinite, so $S + S'$ itself is indefinite.

For purposes of this paper, we define the *index* of a trapezoidal matrix as the number of positive diagonal entries it has. (This coincides with the usual definition of index wherever both definitions apply, namely, to a trapezoidal matrix which is also (symmetric and hence) diagonal. See, for example, [4, Definition 3, p. 92], which is modified in the obvious way to apply over an arbitrary ordered field.) We define the *superindex* (*subindex*) of an arbitrary triangularizable matrix $S$ as the largest (smallest) index of those trapezoidal matrices congruent to $S$. We shall denote by $\sigma(S)$ and $\tau(S)$ the respective superindex and subindex of $S$ (or, sometimes, by $\sigma$ and $\tau$ when $S$ is understood). We must point out that $\sigma(S)$ and $\tau(S)$ may depend on $F$ as well as on $S$. In general, for fixed $S$, we see that "increasing" the field (i.e., extending the field to a larger ordered field) will (weakly) increase $\sigma(S)$ and (weakly) decrease $\tau(S)$.

It follows immediately from definition that if $S$ is a triangularizable matrix and $T$ is a trapezoidal matrix of index $s$ and is congruent to $S$ then

$$0 \leqslant \tau \leqslant s \leqslant \sigma \leqslant r$$

and that there is such a $T$ of index $\sigma$ (and one of index $\tau$). It is also immediate that superindex and subindex (and their domain of definition) are invariant under congruence. The following fact is less immediate, but is still an elementary consequence of the foregoing.

*Fact* 3.3. If $S$ is triangularizable, then ($-S$ is triangularizable and) $\sigma(-S) + \tau(S) = r$ $(= r(S) = r(-S))$.

We shall need one more definition in this section. We have already seen in Fact 1.3 that, when $S$ is regular of rank $r \geqslant 1$, (the congruence class of) $S$ determines a unique congruence class of nonsingular $r \times r$

matrices (the class containing the matrix $A$ of Fact 1.3). Thus all the $(r \times r)$ matrices of this congruence class have nonzero determinants of the same sign. We shall call this sign "the signum of $S$" and denote it by $\delta(S)$ (or sometimes by $\delta$ when $S$ is understood). We define the signum of each (square) zero matrix to be 1. (Thus $\delta$ is defined for all regular matrices. We leave $\delta$ undefined for irregular matrices.) Thus signum (as well as its domain of definition) is invariant under congruence.

We shall need the following two elementary facts (Facts 3.4 and 3.5). The proofs are routine (using earlier results) and so will be omitted.

*Fact* 3.4. If $S$ is regular, then ($-S$ is regular and) $\delta(-S) = (-1)^r \delta(S)$.

*Fact* 3.5. Let $S$ be a regular triangularizable matrix, and let $T$ be a trapezoidal matrix of index $s$ and be congruent to $S$. Then $\delta = (-1)^{r-s}$. Thus in particular $\delta = (-1)^{r-\sigma}$ and hence $\sigma \leqslant r - \frac{1}{2}(1 - \delta)$.

We conclude this section by applying the foregoing concepts to five special cases (mostly well known), all of which are regular and in all of which $\sigma = \tau$ (or $\sigma$ and $\tau$ are undefined). We shall see later (Remark 3 of Section 5) that these five cases are the only ones in which $\sigma = \tau$ (or $\sigma$ and $\tau$ are undefined), that is, are the only ones where the expression "the index of $S$" would have a congruence-invariant meaning (at least over those fields over which Theorem 2 holds).

*Example* 1. $S$ is a nonzero skew matrix. Here $\sigma$ and $\tau$ are undefined (by Fact 1.1), but $S$ is regular and $\delta = 1$ (this follows from the case of [4, Theorem 5–11, pp. 95–96] in which the field is ordered).

*Example* 2. $S$ is symmetric. Here ($S$ is triangularizable and) all the trapezoidal matrices congruent to $S$ are diagonal (since they are symmetric) and by Sylvester's inertia theorem ([4, Theorem 5–7, p. 93], which holds over an arbitrary ordered field) they all have the same index (which is for this reason called the index of $S$). Thus here $\sigma(S) = \tau(S) =$ the index of $S$. Also here $S$ is regular (and $\delta = (-1)^{r-\sigma}$ by Fact 3.5).

*Example* 3. $S + S'$ is nonnegative definite and nonzero. Here there is a trapezoidal matrix $T$ congruent to $S$ by Theorem 1, and, for such $T$, $T + T'$ is congruent to $S + S'$ and hence is nonnegative definite

[4, Theorems 5–1 and 5–2, pp. 86–87]. Thus all the diagonal entries of $T + T'$, which are respectively those of $2T$, are $\geqslant 0$, so $\sigma = \tau = r$. Also, by Fact 3.2 $S$ is regular, and by Fact 3.5 $\delta = 1$.

*Example* 4. $S + S'$ is nonpositive definite and nonzero. Here, following the lines of Example 3, we see that $S$ is triangularizable, $\sigma = \tau = 0$, $S$ is regular, and $\delta = (-1)^r$.

*Example* 5. $S$ is regular with $r = 2$ and $\delta = -1$. Here $S$ is triangularizable (by Theorem 1, as otherwise $\delta$ would $= +1$ by Example 1) and $\sigma = 1$ by Fact 3.5. By Fact 3.4 $\delta(-S) = \delta(S) = -1$, and hence by Fact 3.5 $\sigma(-S) = 1$. Thus by Fact 3.3 $\tau(S) = 1$ (so $\tau = \sigma$).

## 4. EVALUATION OF THE SUPERINDEX

This section is devoted to our second main result, which, under suitable restrictions on $F$, says that, for given rank and given signum (possibly given to be undefined) the superindex is as large as possible, consistent with the results of Section 3.

THEOREM 2. *Let $F$ be an ordered field which contains a square root of each of its positive numbers, and suppose that $S$ is an $n \times n$ nonsymmetric matrix over $F$ and that $S + S'$ is not nonpositive definite. Then* (a) $\sigma \geqslant r - 1$ (*where $\sigma$ is the superindex of $S$ relative to $F$, and $r$ is the rank of $S$*); *more precisely,* (b) $\sigma = r$ *if $S$ is irregular, and* (c) $\sigma = r - \frac{1}{2}(1 - \delta)$ *if $S$ is regular* (*where $\delta$ is the signum of $S$*).

*Proof*. Clearly part (c) follows from part (a) and Fact 3.5. We shall prove parts (a) and (b) by (joint) induction on $n$. (However our proof will amount to separate inductions for parts (a) and (b), because of our case division.) For our induction assertion, $H(n)$, we take the statement of Theorem 2 (minus part (c)). Since $H(1)$ is vacuously true, we shall henceforth suppose that $n \geqslant 2$, and we shall begin our induction with $n = 2$.

For purposes of this proof, we shall call a matrix *standard* trapezoidal provided it is trapezoidal, its first diagonal entry $= 1$, and each of its nonzero diagonal entries has absolute value $= 1$. We shall on several occasions use the obvious fact that if $S$ is standard trapezoidal of index $s$ then the index of $S_1$ is $s - 1$.

Let now $S$ be an arbitrary $n \times n$ nonsymmetric matrix (over $F$) such that $S + S'$ is not nonpositive definite. Then by Fact 3.1 $S$ is congruent (always over $F$) to a trapezoidal matrix whose first diagonal entry is positive. This latter matrix is (diagonally) congruent (by the same reasoning as in the proof of [4, Theorem 5–6, pp. 91–92]) to a standard trapezoidal matrix since $F$ contains a square root of each of its positive numbers. Therefore $S$ is congruent to a standard trapezoidal matrix, and, since $r$, $\sigma$, and $\delta$ (and their respective domains of definition) are invariant under congruence, we may henceforth assume that $S$ itself is standard trapezoidal. In what follows $s$ will always mean the index of $S$. Thus $\sigma \geqslant s \geqslant 1$ (and hence $r \geqslant 1$ also).

When $n = 2$, then $r \leqslant 2$, so $\sigma \geqslant s \geqslant 1 \geqslant r - 1$ (whether or not $S$ is regular), and if $S$ is irregular then $1 \leqslant s \leqslant \sigma \leqslant r < n = 2$ and so $\sigma = r$ $(= 1)$. Thus $H(2)$ is true. In the rest of the proof we thus assume that $n \geqslant 3$ and that $H(n-1)$ is true. We shall also assume that $r \geqslant 2$ (if $r = 1$ we have no more to prove since $1 \leqslant s \leqslant \sigma \leqslant r = 1$ and so $\sigma = r$).

The rest of the proof proceeds by considering various cases for $S$. (There is much overlapping among the cases, but all possibilities are dealt with, in one way or another.) Note that after certain cases further assumptions (the respective contrary assumptions having been dealt with) are added, *which are to apply to all subsequent cases*. (The assumptions we have accumulated so far are that $S$ is $n \times n$, nonsymmetric, and standard trapezoidal of rank $r \geqslant 2$, that $n \geqslant 3$, and that $H(n-1)$ is true.)

Case 1: $S$ is irregular and $S_1$ is irregular. By part (b) of $H(n-1)$, $S_1$ is here congruent to a trapezoidal matrix of index $= r(S_1) = r - 1$. The corresponding (1) subcongruence applied to $S$ gives a matrix which is trapezoidal (by Fact 1.5 or 1.6) and has index $= 1 + (r - 1) = r$. Therefore $\sigma = r$ in this case.

Case 2: $S$ is irregular and $S_1$ is regular and $S_{22} = -1$. Here there is an $i \geqslant r + 1$ for which $S_{i1} \neq 0$. Select such an $i$ and put $y = S_{i1}$ and $u = S_{21}$. Then

$$S[1, 2, i] = \begin{bmatrix} 1 & 0 & 0 \\ u & -1 & 0 \\ y & 0 & 0 \end{bmatrix},$$

in which $y \neq 0$. We next select a nonzero $t \in F$ so that $1 - ut - t^2 > 0$

(the existence of such a $t$ is shown later in this paragraph). Then we apply to $S$ the $[1, 2]$ subcongruence defined by the matrix

$$C = (1 - ut - t^2)^{-1/2}\begin{bmatrix} 1 - ut & t \\ t & 1 \end{bmatrix}$$

and (in view of Fact 1.4) get thereby a matrix $T$ such that

$$T[1, 2, i] = \begin{bmatrix} 1 & 0 & 0 \\ u & -1 & 0 \\ y_1 & x_1 & 0 \end{bmatrix},$$

in which $x_1 = yt(1 - ut - t^2)^{-1/2} \neq 0$. By Fact 1.6 $T$ is therefore standard trapezoidal and so is covered by Case 1. We show that such a $t$ exists by the following considerations, which we shall also need in Case 6 below. For fixed $u$, define the quadratic function $q$ by

$$q(t) \equiv 1 - ut - t^2.$$

The discriminant of $q$ is positive, so $q$ has two zeros, $t_1$ and $t_2$, in $F$. Moreover, $t_1$ and $t_2$ have opposite signs; say $t_1 < 0 < t_2$. Thus

$$q(t) \equiv (t_2 - t)(t - t_1),$$

so for $t_1 < t < t_2$ we have $q(t) > 0$. For present purposes we therefore have only to choose $t$ (strictly) between 0 and $t_2$ (or between $t_1$ and 0).

Case 3: $S$ is irregular and $S_1$ is regular and $S_{22} = 1$. As in Case 2, we here again select an $i \geqslant r + 1$ for which $S_{i1} \neq 0$, and then put $y = S_{i1}$ and $u = S_{21}$. Here

$$S[1, 2, i] = \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ y & 0 & 0 \end{bmatrix},$$

in which $y \neq 0$. Here we apply to $S$ the $[1, 2]$ subcongruence defined by the matrix

$$C = \begin{bmatrix} u & 1 \\ -1 & 0 \end{bmatrix}$$

and (in view of Fact 1.4) get thereby a matrix $T$ such that

$$T[1, 2, i] = \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ uy & y & 0 \end{bmatrix}.$$

By Fact 1.6 $T$ is therefore standard trapezoidal and so is covered by Case 1.

We have now dealt with all the cases in which $S$ is irregular (i.e., we have completed our proof of part (b) of $H(n)$) so henceforth we assume that $S$ is regular (and thus that $S_1$ is also regular) and we (henceforth) need only show that $\sigma \geqslant r - 1$. If $s \geqslant r - 1$ then automatically $\sigma \geqslant s \geqslant r - 1$, so we shall also (henceforth) assume that $s \leqslant r - 2$ (and thus that $r \geqslant 3$, since $s \geqslant 1$).

Case 4: $S_1$ is nondiagonal and $S_1 + S_1'$ is not nonpositive definite. Here $S_1$ is nonsymmetric (since it is trapezoidal and nondiagonal). Thus by part (a) of $H(n-1)$ we have $S_1$ congruent to a trapezoidal matrix of index $\geqslant r(S_1) - 1 = (r-1) - 1 = r - 2$, and when we apply to $S$ the corresponding (1) subcongruence we get (by Fact 1.5 or 1.6) a trapezoidal matrix of index $\geqslant 1 + (r-2) = r - 1$. Therefore $\sigma \geqslant r - 1$ in this case.

Case 5: $S_1$ has a principal submatrix of the form

$$A = \begin{bmatrix} -1 & 0 \\ x & -1 \end{bmatrix} \qquad \text{with} \quad x^2 > 4.$$

Here $A + A'$ is an indefinite submatrix of $S_1 + S_1'$, so $S_1 + S_1'$ is also indefinite. Thus $S$ is here covered by Case 4.

Case 6: $s = 1$ and $S(1, 2 \mid 1, 2] \neq 0$. Here there is an $i \geqslant 3$ such that $S[i \mid 1, 2] \neq 0$ ($i \leqslant r$ since $S$ is regular and trapezoidal). Put $u = S_{21}$, $y = S_{i1}$, and $x = S_{i2}$. Then

$$S[1, 2, i] = \begin{bmatrix} 1 & 0 & 0 \\ u & -1 & 0 \\ y & x & -1 \end{bmatrix},$$

in which $x$ and $y$ are not both zero. We apply to $S$ the $[1, 2]$ subcongruence defined by the matrix $C$ of Case 2 of the present proof, where $t_1 < t < t_2$, but further restrictions on $t$ will be added later. The resulting matrix $T$ is standard trapezoidal by Fact 1.6 since

$$T[1, 2, i] = \begin{bmatrix} 1 & 0 & 0 \\ u & -1 & 0 \\ y_1 & x_1 & -1 \end{bmatrix},$$

where $x_1 = (x + ty)(1 - ut - t^2)^{-1/2}$. We now show that $t$ can be chosen so that $x_1{}^2 > 4$ (this will show that $S$ is congruent to a matrix, namely $T$, covered by Case 5). Namely, if $x$ and $y$ have the same sign or if one of them is zero, we have only to choose $t$ sufficiently near $t_2$ (with $t_1 < t < t_2$) to make $x_1{}^2 > 4$. If $x$ and $y$ have opposite signs, we have only to choose $t$ sufficiently near $t_1$ (with $t_1 < t < t_2$).

Case 7: $s = 1$ and $S_{32} = 0$ and $S_{21} \neq 0$. Here we apply to $S$ the interchanging $[2, 3]$ subcongruence, and get by Fact 1.7 a standard trapezoidal matrix which (by Fact 1.4) is covered by Case 6 since its $(3, 1)$st entry is $S_{21} \neq 0$.

Case 8: $s = 1$ and $S(1, 2 \mid 2] = 0$ and $S(1, 2)$ is nondiagonal. Here $r \geqslant 4$ and there is an $i \geqslant 4$ such that $S[i \mid 3, 4, \ldots, i-1] \neq 0$. Select the *smallest* such $i$. Thus $4 \leqslant i \leqslant r$ and there is a $j$ such that $S_{ij} \neq 0$ and $3 \leqslant j < i$. Therefore $S[3, 4, \ldots, j \mid 2] = 0$ and $S[j \mid 2, 3, \ldots, j-1] = 0$, so by Fact 1.7 the interchanging $[2, j]$ subcongruence applied to $S$ gives a standard trapezoidal matrix which (by Fact 1.4) is covered by Case 6 since its $(i, 2)$nd entry is $S_{ij} \neq 0$.

We have now dealt with all the possibilities for which $s = 1$, so we henceforth assume $s \geqslant 2$ (and hence $r \geqslant 4$). If now $S_1$ is nondiagonal then $S$ is covered by Case 4 (since index $S_1 = s - 1 \geqslant 1$), so we may also (henceforth) assume that $S_1$ is diagonal.

Case 9: $S_{22} = 1$ and $S_{33} = -1$ and $S_{31} \neq 0$. Here we let $u = S_{21}$ and $y = S_{31}$. Then

$$S[1, 2, 3] = \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ y & 0 & -1 \end{bmatrix},$$

in which $y \neq 0$. We apply to $S$ the $[1, 2]$ subcongruence defined by the matrix $C$ of Case 3 (of the present proof) and (in view of Fact 1.4) get thereby a matrix $T$ such that

$$T[1, 2, 3] = \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ uy & y & -1 \end{bmatrix}.$$

By Fact 1.6 $T$ is therefore standard trapezoidal and so is covered by Case 4.

Case 10: There is an $i$ such that $S_{ii} = -1$ and $S_{i1} \neq 0$. Here we pick such an $i$; then $i \geqslant 2$ (since $S$ is standard trapezoidal). First, we apply to $S$ the interchanging $[3, i]$ subcongruence and (by Fact 1.7, since $S_1$ is diagonal) get thus a standard trapezoidal matrix $T$ such that index $T$ = index $S = s$ and $T_{33} = S_{ii} = -1$, and by Fact 1.4 we have that

$T_{31} = S_{i1} \neq 0$ and that $T_1$ is diagonal. Now, if $T_{22} = 1$, then $T$ is covered by Case 9. Otherwise (i.e., if $T_{22} = -1$), there is a $j \geqslant 4$ such that $T_{jj} = 1$ (since index $T \geqslant 2$). Then (by Fact 1.4) applying to $T$ the interchanging $[2, j]$ subcongruence gives us a matrix which is covered by Case 9 (and which is congruent to $T$ and hence to $S$).

Case 11: $S_{22} = 1$ and $S_{33} = -1$ and $S_{31} = 0$ and $S_{21} \neq 0$. Here (as in Case 9) we let $u = S_{21}$ and $y = S_{31}$, and then $S[1, 2, 3]$ is the same as in Case 9, except that here $y = 0$ and $u \neq 0$. Here we apply to $S$ the $[2, 3]$ subcongruence defined by the matrix

$$\frac{1}{3}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

and (by Fact 1.4) get thus a standard trapezoidal matrix $T$ covered by Case 9 since

$$T[1, 2, 3] = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 5u & 3 & 0 \\ 4u & 0 & -3 \end{bmatrix}.$$

Case 12: $S_{i1} = 0$ for every $i$ such that $S_{ii} = -1$. Here (since $S_1$ is diagonal and $S$ is nondiagonal and trapezoidal) there is a $j \geqslant 2$ such that $S_{j1} \neq 0$. We pick such a $j$; then $2 \leqslant j \leqslant r$ (since $S$ is regular) and so $S_{jj} = 1$. We next apply to $S$ the interchanging $[2, j]$ subcongruence and (by Fact 1.7) get thus a standard trapezoidal matrix $T$ such that index $T = s$ and $T_{22} = 1$ (and by Fact 1.4 we have that $T_{21} = S_{j1} \neq 0$ and that $T_1$ is diagonal). Now, if $T_{33} = -1$, then $T$ is covered by Case 11. Otherwise (i.e., if $T_{33} = 1$), there is a $k$ such that $T_{kk} = -1$ (since $s \leqslant r - 2$), and applying to $T$ the interchanging $[3, k]$ subcongruence gives (by Fact 1.4) a matrix which is covered by Case 11 (and which is congruent to $T$ and hence to $S$).

This concludes the proof of the induction step and of Theorem 2.

## 5. CONCLUDING REMARKS

We conclude with several comments on Theorem 2, plus a corollary of it.

*Remark* 1. In the interests of saving space and simplifying the proof of Theorem 2, we did not try to prove it (and its various special cases) under the weakest possible assumptions on $F$. For example, if we only assumed about $F$ that it is *dense* in an ordered field $K$ which contains a square root of each positive element of $K$, then it is fairly clear how to modify the given proof to deal with this weakened hypothesis. (We would say that $F$ is *dense* in $K$ provided that between each two elements of $K$ lies at least one element of $F$.) Thus in particular Theorem 2 would still be true if $F$ is any archimedean (ordered) field, since every archimedean field is isomorphic to a dense subfield of the real field. It is also fairly easy to modify the arguments (in the proof of Theorem 2) suitably to prove the irregular cases or the cases where $r \leqslant 2$ (or, as was done in Example 3 of Section 3, the case where $S + S'$ is nonnegative definite and nonzero) over an arbitrary ordered field $F$. However, it is doubtful if the entire theorem can be proved over an arbitrary ordered field. (The "sticking point," if any, comes in Case 6.)

*Remark* 2. It is not difficult to see that (over a field $F$ dense in a field $K$ containing a square root of each of its positive elements), among trapezoidal matrices of a single congruence class, the index may take on each integer value between the subindex and the superindex of that class, except for the parity restriction (Fact 3.5) in the regular cases. (Unfortunately, this fact is obscured in the given proof of Theorem 2, again in order to avoid lengthening that proof unduly.)

*Remark* 3. It is clear from Theorem 2 and Facts 3.3 and 3.4 that, for given rank and given (possibly undefined) signum, the subindex is as small as possible, consistent with the results of Section 3. More precisely, over any field $F$ for which Theorem 2 holds, $\tau = 0$ if $S$ is irregular and $\tau = \frac{1}{2}[1 - (-1)^r\delta]$ if $S$ is regular and nonsymmetric and $S + S'$ is not nonnegative definite, and consequently $\sigma = \tau$ (or $\sigma$ and $\tau$ are undefined) (i.e., the index is invariant under congruence) only in the five examples at the end of Section 3.

*Remark* 4. We could also define *index* for arbitrary triangular matrices (as the number of positive diagonal entries) and then define *superindex* and *subindex* in terms of this extended definition of index. However, by taking proper precautions in the proof of Fact 1.8 (when $F$ is ordered), we can see that each triangular matrix of index $s$ is congruent to a trapezoidal matrix of index $\geqslant s$ and to a trapezoidal matrix of index $\leqslant s$, so the superindex and subindex would not change under this extension of the definition of index.

Finally, we give here as a corollary of Theorem 2 the following special case of it ($F$ = the real field and $r = n$ and $\delta = 1$), which will be applied elsewhere [1] and is the original motivation for the present paper.

*Corollary*. Let $S$ be a real nonsymmetric $n \times n$ matrix such that $\det S > 0$ and $S + S'$ is not nonpositive definite. Then $S$ is congruent over the real field to a lower triangular matrix all of whose diagonal entries are positive.

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